A Note on the Curve and Surface Contracting Flow

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***Abstract*— In this note, we present a simple approach to calculate the time of blowing up along the curve and surface contracting flow. Our approach dependents on the famous theorem of turning tangents, Gauss-Bonnet Theorem and Willmore Theorem. We also prove that the rotation index of a closed curve and the Euler characteristic of a closed surface are constants along the curve and surface contracting flow.**

***Keywords*— curve contracting flow; surface contracting flow; square mean curvature flow; theorem of turning tangents; Gauss-Bonnet Theorem; Willmore Theorem.**

I. INTRODUCTION AND MAIN RESULTS

In the last few decades, there has been a large amount of literatures on the geometric evolution equation problems, among them, the simplest case is the curve contracting flow in the plane which is established by Gage [1,2], Gage and Hamilton [3] and Grayson [4]. They have proved that a simple closed initial curve remains so and becomes more and more circular during the curve-contracting process, and it collapses to a round point in a finite time.

The curve contracting flow is a family of closed curves:

$$\gamma(x,t): S^1 \times [0,T) \to R^2$$

satisfying the following evolution equation

$$\frac{\partial \gamma(x,t)}{\partial t} = -k(x,t)\vec{n}(x,t), \tag{1}$$

where $k(x,t)$ is the curvature of $\gamma(x,t)$ with respect to the unit inner normal vector $\vec{n}$.

Recall that in [5], Chou established a useful and interesting property for curve contracting flow as follows:

Theorem 1.1. (Chou) *Let γ_0 be a simple closed convex curve in the plane, then the solution $\gamma(x,t)$ of* (1) *with γ_0 as the initial curve exists only in a finite time interval $[0,\omega)$, where*

$$\omega = \frac{S(\gamma_0)}{2\pi},$$

and $S(\gamma_0)$ denotes the area enclosed by the curve γ_0.

The proof of Theorem 1.1 by Chou relies on the Minkowski support function and the maximum principle for evolution equation (1). In this paper, we will give a simple proof of Theorem 1.1 by using the theorem of turning tangents. In fact by the theorem of turning tangents we can also generalize Chou's result for the simple closed convex curve to the general closed curve in the plane:

Theorem 1.2. *Let γ_0 be a closed curve in the plane, then the solution $\gamma(x,t)$ of* (1) *with γ_0 as the initial curve exists only in a finite time interval $[0,\omega_{\gamma_0})$, where*

$$\omega_{\gamma_0} = \frac{S(\gamma_0)}{2\pi\chi(\gamma_0)},$$

and $S(\gamma_0)$ denotes the area enclosed by the curve γ_0. Furthermore, $\chi(\gamma(t))$ is a constant along the curve contracting flow.

In the third part, we consider the Gauss curvature flow, which is a family of closed surfaces $M^2(x,t)$ satisfying the evolution equation as follows:

$$\frac{\partial M^2(x,t)}{\partial t} = -K(x,t)\vec{n}(x,t) \tag{2}$$

where $K(x,t)$ is the Gauss curvature of $M^2(x,t)$ with respect to the unit inner normal vector $\vec{n}$.

As the proof of Theorem 1.2, by using the famous Gauss-Bonnet Theorem, we prove that

Theorem 1.3. *Let* M_0^2 *be a closed surface, then the solution* $M^2(x,t)$ *of* (2) *with* M_0^2 *as the initial surface exists only in a finite time interval* $\left[0,\omega_{M_0^2}\right)$*, where*

$$\omega_{M_0^2}=\frac{V\left(M_0^2\right)}{2\pi\chi\left(M_0^2\right)}.$$

Furthermore, $\chi\left(M^2(t)\right)$ *is a constant along the Gauss curvature flow. In particular, for the sphere* S^2 *,the time of blowing up is*

$$\omega_{S_0^2}=\frac{V\left(S_0^2\right)}{4\pi}.$$

Furthermore, in the fourth part, we deal with the square mean curvature flow, which is a family of closed surfaces $X^2(x,t)$ satisfying the evolution equation as follows:

$$\frac{\partial X^2(x,t)}{\partial t}=-H(x,t)^2\,\vec{n}(x,t) \tag{3}$$

where $H(x,t)$ is the mean curvature of $X^2(x,t)$ with respect to the unit inner normal vector $\vec{n}$.

Then by using the famous Willmore Theorem, we also derive a similar result as follows:

Theorem 1.4. *Let* T_0^2 *be a 2-dim torus in* $\mathbb{R}^3$*, then the solution* $X^2(x,t)$ *of* (3) *with* T_0^2 *as the initial surface exists only in a finite time interval* $\left[0,\omega_{T_0^2}\right)$*, where*

$$\omega_{T_0^2}\le\frac{V\left(T_0^2\right)}{2\pi^2}.$$

The paper is organized as follows. In section 2, we present some preliminaries. In section 3, we prove Theorem 1.2 by using the theorem of turning tangents. In section 4, based on the famous Gauss-Bonnet Theorem we prove Theorem 1.3. In section 5, by Willmore Theorem we prove Theorem 1.4.I believe that our trick can also be appled to other geometric evolution equation with velocity relying on the manifold's topology.

II. PRELIMINARIES

In this section, we recall the famous Theorem of Turning Tangents, Gauss-Bonnet Theorem and Willmore Theorem.

Theorem 2.1 (Theorem of Turning Tangents) *Let* γ *be a closed convex curve in the plane, then the rotation index* $\chi(\gamma)$ *of* γ *is as follows:*

$$\int_\gamma kds=\sum_{i=1}^{m}\left(\tau_i\left(s_i\right)-\tau_i\left(s_{i-1}\right)\right)+\sum_{i=1}^{m}\varphi_i=\pm2\pi\chi(\gamma) \tag{4}$$

Remark 1. For a simple closed curve, the rotation index is ±1, where the sign depends on the orientation of the curve.

Theorem 2.2 (Gauss-Bonnet Theorem) *Let* M^2 *be a closed oriented surface, then*

$$\int_{M^2}KdA=2\pi\chi\left(M^2\right) \tag{5}$$

where K is the Gauss curvature of M^2*, and* $\chi\left(M^2\right)$ *is the Euler characteristic.*

Theorem 2.3 (Willmore Theorem) *Let* T^2 *be a 2-dim torus in* $\mathbb{R}^3$*, then*

$$\int_{T^2}H^2dA\ge2\pi^2 \tag{6}$$

where H is the mean curvature of T^2.

III. PROOF OF THEOREM 1.2

In this section, by using the theorem of turning tangents, we present a simple proof of Theorem 1.2.

Proof of Theorem 1.2. Note that each point (x,t) on the curve $\gamma(x,t)$ moves along the inner normal vector $\vec{n}$ with velocity $k(x,t)$, then for the arch length Δs and in the time interval Δt, the change of the enclosed area is

$$\Delta S=k(x,t)\Delta t\Delta s.$$

Thus the change of enclosed area by the curve $\gamma(t)$ in the time interval Δt is as follows:

$$\Delta S_{\gamma(t)}=\lim_{\max_i|\Delta s_i|\to0}\sum_{i=1}^{m}k_i(x,t)\Delta t\Delta s_i=\int_{\gamma(t)}k(x,t)ds\Delta t$$

By using the theorem of turning tangents and let $\Delta t\to0$, we have

$$\frac{dS(\gamma(t))}{dt} = -\lim_{\Delta t\to 0}\frac{\Delta S_{\gamma(t)}}{\Delta t} = -\lim_{\Delta t\to 0}\frac{\int_{\gamma(t)} k(x,t)\,ds\Delta t}{\Delta t}$$
$$= -\int_{\gamma(t)} k(x,t)\,ds = -2\pi\chi(\gamma(t)) \tag{7}$$

Hence at time *t*, the area $S(\gamma(t))$ enclosed by the closed curve $\gamma(t)$ is

$$S(\gamma(t)) = S(\gamma_0) - 2\pi\int_0^t \chi(\gamma(t))dt .$$

Since $\chi(\gamma(t))$ is continue with t along the curve contracting flow and its value is an integer, it follows that for any time $t>0$ we have

$$\chi(\gamma(t)) \equiv \chi(\gamma_0) .$$

Thus

$$S(\gamma(t)) = S(\gamma_0) - 2\pi\chi(\gamma_0)t ,$$

and the time of blowing up along the curve contracting flow (1) is

$$\omega_{\gamma_0} = \frac{S(\gamma_0)}{2\pi\chi(\gamma_0)} .$$

In particular, for the simple closed convex curve, the rotation index is ± 1, thus we have

$$S(\gamma(t)) = S(\gamma_0) - 2\pi t ,$$

and the time of blowing up is

$$\omega = \frac{S(\gamma_0)}{2\pi} .$$

IV. PROOF OF THEOREM 1.3

In this section, as the proof of Theorem 1.2, we also present a simple proof of Theorem 1.3 by using the Gauss-Bonnet Theorem.

Proof of Theorem 1.3. Note that each point (x,t) on the surface $M^2(x,t)$ moves along the inner normal vector $\vec{n}$ with velocity is the Causs curvature $K(x,t)$, then for the area ΔS and in the time interval Δt, the change of enclosed volume ΔV is

$$\Delta V = K(x,t)\Delta t\Delta S$$

Thus the change of enclosed volume by the surface $M^2(t)$ in the time interval Δt is as follows:

$$\Delta V_{M^2(t)} = \lim_{\max_i |\Delta S_i|\to 0}\sum_{i=1}^{m} K_i(x,t)\Delta t\Delta S_i = \int_{M^2(t)} K(x,t)\,dA\Delta t$$

By using the Gauss-Bonnet Theorem and let $\Delta t \to 0$, we have

$$\frac{dV(M^2(t))}{dt} = -\lim_{\Delta t\to 0}\frac{\Delta V_{M^2(t)}}{\Delta t} = -\lim_{\Delta t\to 0}\frac{\int_{M^2(t)} KdA\Delta t}{\Delta t}$$
$$= -\int_{M^2(t)} KdA = -2\pi\chi(M^2(t)) \tag{8}$$

Hence at time *t*, the volume $V(M^2(t))$ enclosed by the closed surface $M^2(t)$ is

$$V(M^2(t)) = V(M_0^2) - 2\pi\int_0^t \chi(M^2(t))dt$$

Since $\chi(M^2(t))$ is continue with *t* along the the Gauss curvature flow and its value is an integer, it follows that for any time $t>0$ we have

$$\chi(M^2(t)) \equiv \chi(M_0^2) .$$

Thus

$$V(M^2(t)) = V(M_0^2) - 2\pi\chi(M_0^2)t ,$$

and the time of blowing up along the Gauss curvature flow (2) is

$$\omega_{M_0^2} = \frac{V(M_0^2)}{2\pi\chi(M_0^2)} .$$

In particular, for the sphere S^2, we have the Euler characteristic is 2, thus we have

$$V(S^2(t)) = V(S_0^2) - 4\pi t ,$$

and the time of blowing up is

$$\omega_{S_0^2} = \frac{V(S_0^2)}{4\pi} .$$

V. PROOF OF THEOREM 1.4

In this section, as the proof of Theorem 1.2 and 1.3, we present the proof of Theorem 1.4 by using the Willmore Theorem.

Proof of Theorem 1.4. Note that each point (x,t) on the surface $X^2(x,t)$ moves along the inner normal

vector $\vec{n}$ with velocity is the square mean curvature $H(x,t)^2$, then for the area ΔS and in the time interval Δt, the change of enclosed volume ΔV is

$$\Delta V = H(x,t)^2 \Delta t \Delta S$$

Thus the change of enclosed volume by the surface $X^2(t)$ in the time interval Δt is as follows:

$$\Delta V_{X^2(t)} = \lim_{\max_i |\Delta S_i| \to 0} \sum_{i=1}^{m} H_i(x,t)^2 \Delta t \Delta S_i = \int_{X^2(t)} H(x,t)^2 \, dA \Delta t$$

By using the Willmore Theorem and let $\Delta t \to 0$, we have

$$\frac{dV\left(X^2(t)\right)}{dt} = -\lim_{\Delta t \to 0} \frac{\Delta V_{X^2(t)}}{\Delta t} = -\lim_{\Delta t \to 0} \frac{\int_{X^2(t)} H^2 dA \Delta t}{\Delta t}$$
$$= -\int_{X^2(t)} H^2 dA \le -2\pi^2 \quad . \qquad (9)$$

Hence at time t, the volume $V\left(X^2(t)\right)$ enclosed by the closed surface $X^2(t)$ is

$$V\left(X^2(t)\right) \le V\left(T_0^2\right) - 2\pi^2 t \, .$$

Thus the time of blowing up along the square mean curvature flow (3) satisfies

$$\omega_{T_0^2} \le \frac{V\left(T_0^2\right)}{2\pi^2} \, .$$

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